Exercise 1.5.12

Assume that the temperature is spherically symmetric, $u = u(r,t)$, where r is the distance from a fixed point ($r^2 = x^2 + y^2 + z^2$). Consider the heat flow (without sources) between any two concentric spheres of radii a and b.

(a) Show that the total heat energy is $4\pi \int_a^b c\rho u r^2\, dr$.

(b) Show that the flow of heat energy per unit time out of the spherical shell at $r = b$ is $-4\pi b^2 K_0 \partial u/\partial r\big|_{r=b}$. A similar result holds at $r = a$.

(c) Use parts (a) and (b) to derive the spherically symmetric heat equation

$$\frac{\partial u}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right).$$

Solution

The law of conservation of energy states that energy is neither created nor destroyed. If some amount of thermal energy enters a spherical shell at $r = a$, then that same amount must exit at $r = b$ for the temperature to remain the same. If more (less) thermal energy enters at $r = a$ than exits at $r = b$, then the amount of thermal energy in the shell will change, leading to an increase (decrease) in its temperature. The mathematical expression for this idea, an energy balance, is as follows.

$$\text{rate of energy in} - \text{rate of energy out} = \text{rate of energy accumulation}$$

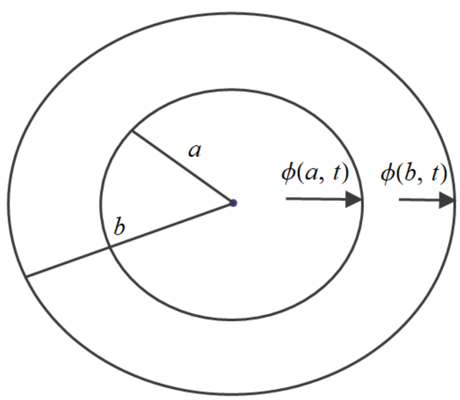

Figure 1: This is a schematic of the spherical shell that the thermal energy flows through (integral formulation). It flows in at $r = a$ and out at $r = b$.

The flux is defined to be the rate that thermal energy flows through the shell per unit area, and we denote it by $\phi = \phi(r,t)$. Multiplying it by the cross-sectional area gives the rate of energy flow. If we let U represent the amount of energy in the shell, then the energy balance over it is

$$A(a)\phi(a,t) - A(b)\phi(b,t) = \left.\frac{dU}{dt}\right|_{\text{shell}}.$$

Factor a minus sign from the left side.

$$-[A(b)\phi(b,t) - A(a)\phi(a,t)] = \left.\frac{dU}{dt}\right|_{\text{shell}}$$

By the fundamental theorem of calculus, the term in square brackets is an integral.

$$-\int_a^b \frac{\partial}{\partial r}[A(r)\phi(r,t)]\,dr = \left.\frac{dU}{dt}\right|_{\text{shell}}$$

The thermal energy in the shell is obtained by integrating the thermal energy density $e(r,t)$ over the shell's volume V.

$$-\int_a^b \frac{\partial}{\partial r}[A(r)\phi(r,t)]\,dr = \frac{d}{dt}\int_V e\,dV$$

For a nonuniform sphere with spherically symmetric mass density $\rho(r)$, specific heat $c(r)$, and temperature $u(r,t)$, the thermal energy density is the product $\rho(r)c(r)u(r,t)$.

$$-\int_a^b \frac{\partial}{\partial r}[A(r)\phi(r,t)]\,dr = \frac{d}{dt}\int_V \rho(r)c(r)u(r,t)\,dV$$

A spherical shell has a volume differential $dV = A(r)\,dr = 4\pi r^2\,dr$, where $A(r)$ is the surface area at r. The volume integral turns into one over the radius.

$$-\int_a^b \frac{\partial}{\partial r}[4\pi r^2\phi(r,t)]\,dr = \frac{d}{dt}\int_a^b \rho(r)c(r)u(r,t)(4\pi r^2\,dr)$$

Therefore, the total thermal energy in the shell is

$$\boxed{U = 4\pi\int_a^b \rho(r)c(r)u(r,t)r^2\,dr.}$$

Divide both sides of the energy balance by 4π and bring the minus sign and time derivative inside the integrals they're in front of.

$$\int_a^b \left\{-\frac{\partial}{\partial r}[r^2\phi(r,t)]\right\}dr = \int_a^b \rho(r)c(r)\frac{\partial u}{\partial t}r^2\,dr$$

Since the two integrals are equal over the same interval of integration, the integrands must be equal.

$$-\frac{\partial}{\partial r}[r^2\phi(r,t)] = \rho(r)c(r)\frac{\partial u}{\partial t}r^2$$

According to Fourier's law of conduction, the heat flux is proportional to the temperature gradient.

$$\phi = -K_0(r)\frac{\partial u}{\partial r},$$

where $K_0(r)$ is a proportionality constant known as the thermal conductivity. It varies as a function of r because the sphere is nonuniform. As a result, the energy balance becomes an equation solely for the temperature.

$$-\frac{\partial}{\partial r}\left\{-r^2K_0(r)\frac{\partial u}{\partial r}\right\} = \rho(r)c(r)\frac{\partial u}{\partial t}r^2$$

Dividing both sides by r^2, we find that the equation for the temperature in a nonuniform sphere is

$$\rho(r)c(r)\frac{\partial u}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 K_0(r)\frac{\partial^2 u}{\partial r^2}\right].$$

If we assume that the sphere is uniform, then ρ, c, and K_0 are constant, and the equation simplifies to

$$\rho c\frac{\partial u}{\partial t} = \frac{K_0}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial^2 u}{\partial r^2}\right).$$

Divide both sides by ρc and set $k = K_0/\rho c$ to obtain the spherically symmetric heat equation for a uniform sphere.

$$\boxed{\frac{\partial u}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)}$$

With Fourier's law in hand, the rate of thermal energy flowing at $r = a$ and $r = b$ can be calculated.

$$\text{Rate of Thermal Energy Flowing at } r = a: \quad A(a)\phi(a,t) = 4\pi a^2\left[-K_0(a)\frac{\partial u}{\partial r}(a,t)\right]$$

$$\text{Rate of Thermal Energy Flowing at } r = b: \quad A(b)\phi(b,t) = 4\pi b^2\left[-K_0(b)\frac{\partial u}{\partial r}(b,t)\right]$$

Therefore,

$$\boxed{\text{Rate of Thermal Energy Flowing at } r = a: \quad -4\pi a^2 K_0(a)\frac{\partial u}{\partial r}(a,t)}$$

and

$$\boxed{\text{Rate of Thermal Energy Flowing at } r = b: \quad -4\pi b^2 K_0(b)\frac{\partial u}{\partial r}(b,t).}$$